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Magnetotransport in the 2D Lorentz model: linear and nonlinear effects of a weak electric field

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Abstract. We study the two-dimensional (2D) classical Lorentz model in a transverse magnetic and an in-plane electric field, in the regime where the dimensionless electric field is smaller than all other parameters. Since, therefore, the dimensionless density must be kept finite, we start from the Liouville equation and derive, by the multiple time scale method, the equations governing nonlinear as well as linear transport. The same diffusion tensor, formally rederived as the Kubo expression is, surprisingly, found to govern both regimes, albeit in different manner. Subsequently, explicit asymptotic results for the two components of the current density are calculated in the low-density regime.

1. Introduction

The classical Lorentz model [1] has now been studied for over 90 years. In this model non-interacting point particles ('electrons' with charge $-e$) move in a random array of stationary scatterers of short range. Despite its long history, this seemingly simple model continues to reveal unexpected properties. In particular, it was recently realized [2] that in two dimensions (2D), and in a transverse magnetic field $\mathcal{B}$, the Boltzmann equation is *not* the correct kinetic equation for the Lorentz model, even in the Grad limit [3]. In this limit the number density of the scatterers, $n \to \infty$, while their radius, $a \to 0$, in such a way that the mean-free path, Λ, stays constant, and the dimensionless density $\eta = na^2 \to 0$. In the Grad limit the 2D Lorentz model in a magnetic field is governed by a *generalized* Boltzmann equation, with a collision operator that is local in space, but with an interesting non-Markovian structure. Moreover, it was pointed out in [2] that qualitatively new problems arise if an in-plane electric field is added.

On the other hand, the Lorentz model in an electric field $\mathcal{E}$, with *no* magnetic field present was, on the basis of the Boltzmann equation, thoroughly studied some time ago [4, 5]. Its properties are interesting. A weak electric field will, after an initial transient described by kinetic theory, for a sizeable time interval distort the initial equilibrium distribution and generate a current density in the manner predicted by linear response theory. On this time scale, standard linear results (Ohm's law) are valid. However, since the model contains no dissipative mechanism, and since the particles preferentially move in the direction favoured by the electric field, the energy of the charged particles will, on a longer time scale, slowly grow. The time dependence of this process was examined in [4, 5]. Asymptotically, the current was found to be proportional to $\mathcal{E}^{1/3}$, and to decay as $t^{-1/3}$.

The present paper contains the first study of the 2D Lorentz model in which *both* fields are present, a transverse magnetic, and an in-plane electric field. We do not discuss its properties for arbitrary field strengths and densities here. The model is expected to behave quite differently, depending on the magnitude chosen for the various dimensionless parameters. In particular, it was argued in [2] that in a transverse $\mathcal{B}$-field, the diffusion tensor is singular as the $\mathcal{E} \to 0$ in the Grad limit. In this paper we focus on the regime in which the dimensionless electric field is small with respect to *all* other parameters. As a consequence we cannot, in principle, take the Grad limit from the start, but must let the dimensionless density be finite. The only reliable basis from which to proceed, under these circumstances, is the Liouville equation or, equivalently, the hierarchy.

Here we choose to start from the Liouville equation. From this most fundamental of starting points, we shall primarily be interested in the 'hydrodynamic' time scale(s), on which there is macroscopic transport, linear and nonlinear. The multiple time scale method is particularly well suited to a systematic study of problems of this sort. This method can be traced back to work by Krylov and Bogolyubov in the 1930s [6]. A particularly clear account of the method can be found in [7], and its relevance for classical kinetic theory is discussed in [8]. It has already been applied by one of the authors to the Lorentz model in an electric field [9]. We shall adopt this subtle and efficient method here, and keep our discussion general. The emphasis on systematics and generality will pay off, as we shall see.

The initial transient (on what we shall call the τ_0-scale) from the equilibrium distribution at $t = 0$ can only be described in detail by kinetic theory. Since the dimensionless density of scatterers is taken to be *finite*, the systematic construction of such a theory is a formidable task [10], and we shall bypass it here. The next time scale (the τ_1-scale) is that of linear response. Finally, the energy of the moving particles will, on average, grow in time (on the τ_2-scale). With our fundamental starting point, we derive the 'hydrodynamic' equations governing the last two time scales, including exact formal expressions for the appropriate transport coefficients. It should come as no surprise that, on the τ_1-scale, we rederive the Kubo formula for the diffusion tensor, and the Nernst–Einstein relation by which the conductivity tensor follows. What is less obvious is that it is the *same* diffusion tensor which governs, albeit in a different manner, nonlinear transport on the τ_2-scale. This fundamental result, found because we insisted on a general approach has, to our knowledge, not been noticed before.

Nevertheless, we shall see in retrospect that the diffusion equation on the energy axis, from which the nonlinear role of the diffusion tensor follows, can be found directly by a simple physical argument.

In order to provide a concrete illustration of our general results, we use the diffusion tensor given by the generalized Boltzmann equation as an approximate representation of the true diffusion tensor at small, but finite, dimensionless densities. We calculate the current densities on the hydrodynamic time scales, in particular the initial and the asymptotic behaviour for long times. Since, asymptotically, the magnetic field is effectively turned off, we can make contact with the previously found asymptotic results [4, 5], mentioned above. In addition, the asymptotic behaviour of the current density transverse to the electric field is calculated.

We have organized the paper as follows: After some basic material has been presented in section 2, we carry out the multiple time scale analysis in section 3, with some details relegated to the appendix. General results for linear and nonlinear transport are presented in section 4. Explicit low-density results are discussed in section 5, and concluding remarks constitute section 6.

2. Basics

For simplicity, all calculations will be made for hard disc scatterers with radius a. As will become clear, some of our basic results should be valid for more general models. It is convenient already from the beginning to introduce dimensionless variables. Lengths will be scaled by the mean-free path $\Lambda = (2na)^{-1}$, time by the (inverse) cyclotron frequency $\omega = e\mathcal{B}/m$ (with e the elementary charge), and velocities by the speed v_0. This speed could be a given initial speed of the electron, or it could be one characterizing an initial distribution, e.g., the Maxwell distribution which would give $v_0 = \sqrt{2k_\mathrm{B}T/m}$, with T the absolute temperature. In other words, we write

$$
\begin{aligned}
&\boldsymbol{r} = \Lambda \boldsymbol{x} && \text{position of the electron}\\
&\boldsymbol{R}_i = \Lambda \boldsymbol{X}_i && \text{position of scatterer } i,\ i = 1, 2, \ldots, N\\
&t = \omega^{-1}\tau && \text{time}\\
&\boldsymbol{v} = v_0 \boldsymbol{u} && \text{velocity of the electron.}
\end{aligned}
\tag{1}
$$

In addition, we introduce the dimensionless ratios

$$
\begin{aligned}
&\eta = a/(2\Lambda) = na^2 && \text{dimensionless number density of scatterers}\\
&\varepsilon = v_\mathrm{d}/v_0 = (\mathcal{E}/\mathcal{B})/v_0 && \text{ratio of drift velocity to } v_0\\
&\alpha = v_0/(\omega\Lambda) = R_\mathrm{c}/\Lambda && \text{ratio of initial cyclotron radius to } \Lambda.
\end{aligned}
\tag{2}
$$

The joint distribution of one moving electron and N hard disc scatterers we write in scaled variables as $\rho(e, 1, \ldots, N; \tau)$, where $e \equiv (\boldsymbol{x}, \boldsymbol{u}) = (\boldsymbol{x}, u, \phi)$ specifies the dynamic state of the electron ((u, ϕ) are the polar coordinates of the velocity), and $(1, \ldots, N) \equiv (\boldsymbol{X}_1, \ldots, \boldsymbol{X}_N)$ the positions of the scatterers. The normalization is $\int \mathrm{d}e\, \mathrm{d}1 \cdots \mathrm{d}N\, \rho(e, 1, \ldots, N; \tau) = 1$. In terms of these variables the Liouville equation reads

$$
\left[\frac{\partial}{\partial\tau} + \alpha\boldsymbol{u}\cdot\frac{\partial}{\partial\boldsymbol{x}} + \frac{\partial}{\partial\phi} - \varepsilon\hat{\mathcal{E}}\cdot\frac{\partial}{\partial\boldsymbol{u}}\right]\rho(e, 1, \ldots, N; \tau)
$$

$$
= 2\alpha\eta\sum_{j=1}^{N} T(e, j)\rho(e, 1, \ldots, N; \tau) \tag{3}
$$

where $T(e, j)$ is the binary collision operator involving scatterer j:

$$
\begin{aligned}
T(e, j) &= u\int \mathrm{d}\hat{\sigma}\ (\hat{\sigma}\cdot\hat{u})[\theta(\hat{\sigma}\cdot\hat{u})b_{\hat{\sigma}} + \theta(-\hat{\sigma}\cdot\hat{u})]\delta(\boldsymbol{x} - \boldsymbol{X}_j - 2\eta\hat{\sigma})\\
b_{\hat{\sigma}}\,\chi(\boldsymbol{u}) &= \chi\left(\boldsymbol{u} - 2(\hat{\sigma}\cdot\boldsymbol{u})\hat{\sigma}\right).
\end{aligned}
\tag{4}
$$

Here $\delta(\cdot)$ and $\theta(\cdot)$ are the Dirac delta and unit step functions, respectively. Unit vectors are distinguished by a hat, $\hat{\sigma}$ is the unit vector from the centre of the scatterer in the direction of the point of collision, and $\int \mathrm{d}\hat{\sigma}$ denotes the angular integral.

Before proceeding, note the following identity which, physically, corresponds to particle conservation:

$$
\int \mathrm{d}\hat{u}\ T(e, j)F(e) = 0 \tag{5}
$$

with $F(e)$ an arbitrary function of $e = (\boldsymbol{x}, \boldsymbol{u})$. The proof is straightforward. Insert (4) into (5). Change variable $\boldsymbol{u} \to \boldsymbol{w} = \boldsymbol{u} - 2(\hat{\sigma}\cdot\boldsymbol{u})\hat{\sigma}$, in the first (gain) term, and use that $\hat{\sigma}\cdot\hat{u} = -\hat{\sigma}\cdot\hat{w}$. This shows that the integrated gain term equals the integrated loss term for arbitrary $F(e)$, and the identity (5) follows immediately.

From here on we choose, for simplicity, the distribution of the scatterers to be completely random, i.e. with no penalty for overlaps. This is not in conflict with the fact that, with respect to the electron, the scatterers act as hard discs. Incorporating this feature into the initial distribution $\rho(e, 1, \ldots, N; 0)$, we write

$$\rho(e, 1, \ldots, N; 0) = f(e; 0) \prod_{j=1}^{N} \left[\frac{\theta(|\boldsymbol{x} - \boldsymbol{X}_j| - 2\eta)}{A - 4\pi\eta^2} \right] \tag{6}$$

where $f(e; \tau)$ is the reduced distribution function for the electron, and A is the total area of the system on the scale of Λ^2. Basically, it is the time evolution of the electron distribution function that we are interested in, so we integrate the Liouville equation over the positions of all the scatterers, and go to the thermodynamic limit, $\lim_\infty$: $N \to \infty,\ A \to \infty,\ N/A = n\Lambda^2 = (4\eta)^{-1} =$ constant. We then arrive at the equation

$$\left[\frac{\partial}{\partial\tau} + \alpha \boldsymbol{u} \cdot \frac{\partial}{\partial \boldsymbol{x}} + \frac{\partial}{\partial\phi} - \varepsilon \hat{\mathcal{E}} \cdot \frac{\partial}{\partial \boldsymbol{u}} \right] f(e; \tau)$$
$$= 2\eta\alpha \lim_\infty N \int \mathrm{d}1 \cdots \mathrm{d}N\ T(e, 1)\rho(e, 1, \ldots, N; \tau). \tag{7}$$

This equation depends on the three dimensionless variables ε, ρ, and α. In a spatially homogeneous setting, to which we shall specialize shortly, the two last ones appear as a product only. We are primarily interested in studying the behaviour of the system in various electric field regimes and, typically, with $\eta \ll 1$, i.e. close to the Grad limit, defined as $n \to \infty, a \to 0, 2na = \Lambda^{-1} =$ constant. The simplest regime is that in which ε is considered small compared with every other parameter. Physically, this corresponds to the regime where the electric field is sufficiently weak that the drift during one cyclotron revolution is much smaller than the radius of the scatterer, $(2\pi/\omega)(\mathcal{E}/\mathcal{B}) \ll a$. This is the regime which we study by the multiple time scale technique in the following section.

3. The multiple time scale technique

In the multiple time scale technique one replaces the single time variable τ by a set of times $\tau_0, \tau_1, \tau_2, \ldots$, which in the distribution $f^\varepsilon(e; \tau_0, \tau_1, \tau_2, \ldots)$ are treated as independent variables. Similarly, the derivative with respect to τ is expanded as

$$\frac{\partial}{\partial\tau} \longrightarrow \frac{\partial}{\partial\tau_0} + \varepsilon \frac{\partial}{\partial\tau_1} + \varepsilon^2 \frac{\partial}{\partial\tau_2} + \cdots . \tag{8}$$

The *physical* subspace in this multidimensional extension is the *line* defined by

$$\tau_0 = \tau; \quad \tau_1 = \varepsilon\tau; \quad \tau_2 = \varepsilon^2\tau; \quad \ldots \tag{9}$$

so that

$$f(e; \tau) = f^\varepsilon(e; \tau, \varepsilon\tau, \varepsilon^2\tau, \ldots). \tag{10}$$

The advantage of this extension to multiple time space is that when we expand amplitudes in standard fashion, say,

$$f^\varepsilon(e; \tau_0, \tau_1, \ldots) = f^{(0)}(e; \tau_0, \tau_1, \ldots) + \varepsilon f^{(1)}(e; \tau_0, \tau_1, \ldots) + \cdots \tag{11}$$

we now have the freedom to *require* that each term in the amplitude expansion remains finite on *every* time scale. In this way we remove secular divergences and can let any $\tau_i \to \infty$ with impunity. So far these statements constitute nothing but an optimistic general program. It remains to show that the program actually works in our context. Note that the expansion (11) results from integration of the analogous expansion

$$\rho^{\varepsilon}(e, 1, \ldots, N; \tau_0, \tau_1, \ldots) = \rho^{(0)}(e, 1, \ldots, N; \tau_0, \tau_1, \ldots) + \varepsilon\rho^{(1)}(e, 1, \ldots, N; \tau_0, \tau_1, \ldots) + \cdots \tag{12}$$

over the positions of the scatterers.

3.1. Zeroth order

From now on we restrict the analysis to the spatially homogeneous case. We therefore write $f(e;\tau) = f(\boldsymbol{u};\tau)/A$, with normalization $\int \mathrm{d}\boldsymbol{u}\, f(\boldsymbol{u};\tau) = 1$. The electric field is turned on at $t = 0$. It is convenient to choose the state at $t = 0$ to be an equilibrium state in the magnetic field. As a consequence, $f(u; 0)$ is assumed to be rotationally invariant. *Any* distribution with this property is an equilibrium one-particle distribution for a spatially homogeneous Lorentz model in a magnetic field. With these stipulations, and to zeroth order in ε, equation (7) reduces to

$$\left[\frac{\partial}{\partial\tau_0} + \frac{\partial}{\partial\phi}\right] f^{(0)}(\boldsymbol{u}; \tau_0, \tau_1, \ldots) = 2\alpha\eta \lim_{\infty} N \int \mathrm{d}\boldsymbol{x} \int \mathrm{d}1 \cdots \mathrm{d}N\ T(e, 1)\rho^{(0)}(e, 1, \ldots, N; \tau_0, \tau_1, \ldots). \tag{13}$$

We adopt the convention that the *entire* initial condition $\rho(e, 1, \ldots, N; \tau = 0)$ is carried by the zeroth-order term $\rho^{(0)}(e, 1, \ldots, N; \tau_0 = 0, \tau_1 = 0, \ldots)$. Introduce the operator $P = (2\pi)^{-1} \int \mathrm{d}\phi$ which, acting on any function of u, projects out the rotationally invariant part. From the identity (5) one concludes that P acting on (13) gives $\partial P f^{(0)}/\partial\tau_0 = 0$.

We also need a statement on the dependence of the complement $Qf^{(0)} = (1 - P)f^{(0)}$ on τ_0. It is then convenient to return to the zeroth-order Liouville equation, which reads

$$\left[\frac{\partial}{\partial\tau_0} + \alpha\boldsymbol{u}\cdot\frac{\partial}{\partial\boldsymbol{x}} + \frac{\partial}{\partial\phi}\right]\rho^{(0)}(e, 1, \ldots, N; \tau_0, \tau_1, \ldots) = 2\alpha\eta \sum_{j=1}^{N} T(e, j)\rho^{(0)}(e, 1, \ldots, N; \tau_0, \tau_1, \ldots). \tag{14}$$

The physical initial state, with $f(\boldsymbol{x}, \boldsymbol{u}; 0) = f(u; 0)/A$, follows from (6) as

$$\rho(e, 1, \ldots, N; 0) = \frac{f(u; 0)}{A} \prod_{j=1}^{N} \frac{\theta(|\boldsymbol{x} - \boldsymbol{X}_j| - 2\eta)}{A - 4\pi\eta^2}. \tag{15}$$

With the choice (15), f is clearly rotationally invariant initially. The question we face is whether collisions can break this invariance. For an arbitrary initial state, they can†. However, with the initial state (15) chosen here, the evolution in τ_0, governed by (14), leaves $\rho^{(0)}$ rotationally invariant. The reason is the following one. The zeroth-order Liouville equation (14) contains no electric field. In that context, the initial state is an *equilibrium state*: It is rotationally invariant in velocity space. In addition, the $(N + 1)$-particle density is *constant* for all configurations, except the forbidden ones, in which the

† This can be illustrated by the following example. Let all scatterers be distributed in the right half-plane, but with the moving particle initially in the left half-plane. Clearly, an initially isotropic velocity distribution will, in this case, develop into a non-isotropic one as a result of collisions.

electron overlaps with at least one scatterer (the scatterers, on the other hand, are allowed mutually to overlap freely). In particular, the density is independent of the location of the electron relative to the external boundary of the system. Thus, with the initial state being an equilibrium distribution with respect to the zeroth-order Hamiltonian, we conclude that $\rho^{(0)}$, at $\tau_1 = \tau_2 = \cdots = 0$, must be independent† of τ_0. With the physical initial state (15) as initial data for $\rho^{(0)}$, we have then shown that, for arbitrary τ_0,

$$\rho^{(0)}(e, 1, \ldots, N; \tau_1 = 0, \tau_2 = 0, \ldots) = \frac{f(u; 0)}{A} \prod_{j=1}^{N} \frac{\theta(|\boldsymbol{x} - \boldsymbol{X}_j| - 2\eta)}{A - 4\pi\eta^2}. \tag{16}$$

In order to generalize (16) to arbitrary $\tau_i, \ i \neq 0$, we now use the requirement that no secular divergences should appear in our expansion. In particular, $\rho^{(0)}$ should be well defined as $\tau_0 \to \infty$. This implies that

$$\lim_{\tau_0 \to \infty} \frac{\partial}{\partial \tau_0} \rho^{(0)}(e, 1, \ldots, N; \tau_0, \tau_1, \ldots) = 0. \tag{17}$$

As a result, equation (14) gives

$$\left[\alpha \boldsymbol{u} \cdot \frac{\partial}{\partial \boldsymbol{x}} + \frac{\partial}{\partial \phi} - 2\alpha\eta \sum_{j=1}^{N} T(e, j)\right] \rho^{(0)}(e, 1, \ldots, N; \infty, \tau_1, \ldots) = 0 \tag{18}$$

which demonstrates that $\rho^{(0)}(e, 1, \ldots, N; \infty, \tau_1, \ldots)$ is an *equilibrium* state. The converse of the argument given after (15) then shows that this state must have been an equilibrium state for *all* τ_0. That is, for arbitrary $\tau_i, \ i \neq 0$, the state must be of the form

$$\rho^{(0)}(e, 1, \ldots, N; \tau_1, \tau_2, \ldots) = \frac{f^{(0)}(u; \tau_1, \tau_2, \ldots)}{A} \prod_{j=1}^{N} \frac{\theta(|\boldsymbol{x} - \boldsymbol{X}_j| - 2\eta)}{A - 4\pi\eta^2} \tag{19}$$

with $f^{(0)}(u; \tau_1, \tau_2, \ldots)$ being some rotationally invariant function satisfying the condition $f^{(0)}(u; 0, 0, \ldots) = f(u; 0)$. This state, then, we adopt as the zeroth-order $(N+1)$-particle density in our multiple time scale scheme. It provides a stationary solution to the zeroth-order Liouville equation (14), with initial condition consistent with (15).

From the independence of $\rho^{(0)}$ on τ_0, we immediately conclude that, for *arbitrary* $\tau_1, \tau_2, \ldots$,

$$\frac{\partial f^{(0)}}{\partial \tau_0} = 0. \tag{20}$$

3.2. *First order*

To first order the Liouville equation becomes

$$\left(\frac{\partial}{\partial \tau_1} - \hat{\mathcal{E}} \cdot \frac{\partial}{\partial \boldsymbol{u}}\right) \rho^{(0)} + \left(\frac{\partial}{\partial \tau_0} + \alpha \boldsymbol{u} \cdot \frac{\partial}{\partial \boldsymbol{x}} + \frac{\partial}{\partial \phi}\right) \rho^{(1)} = 2\alpha\eta \sum_{j=1}^{N} T(e, j) \rho^{(1)}. \tag{21}$$

Integration and passage to the thermodynamic limit in the spatially homogeneous case gives the corresponding equation for $f^{(1)}$:

$$\left(\frac{\partial}{\partial \tau_1} - \hat{\mathcal{E}} \cdot \hat{u} \frac{\partial}{\partial u}\right) f^{(0)}(u; \tau_1, \tau_2, \ldots) + \left(\frac{\partial}{\partial \tau_0} + \frac{\partial}{\partial \phi}\right) f^{(1)}(\boldsymbol{u}; \tau_0, \tau_1, \ldots)$$
$$= 2\alpha\eta \lim_{\infty} N \int \mathrm{d}\boldsymbol{x} \int \mathrm{d}1 \cdots \mathrm{d}N \; T(e, 1) \rho^{(1)}(e, 1, \ldots, N; \tau_0, \tau_1, \ldots). \tag{22}$$

† This can also be shown by a direct, but somewhat technical, calculation based on (14) with initial state (15).

Now act on (22) with the projection operator P. In view of the identity (5), this causes the right-hand side to vanish. On the left-hand side, only $(\partial/\partial\tau_1)f^{(0)}$ and $(\partial/\partial\tau_0)f^{(1)}$ survive, in principle, the projection. However, if $(\partial/\partial\tau_1)f^{(0)}$ were finite, this would cause a secular divergence in $Pf^{(1)}$ as $\tau_0 \to \infty$. We *require* that such secular divergences do not exist and, accordingly, we insist that $f^{(0)}$ also remains constant on the τ_1 time scale. Thus we have

$$f^{(0)} = f^{(0)}(u; \tau_2, \tau_3, \ldots) \qquad Pf^{(1)} = Pf^{(1)}(\boldsymbol{u}; \tau_1, \tau_2, \ldots). \tag{23}$$

Note that it is *only* the angular average of $f^{(1)}$ which has been shown to be independent of τ_0.

However, we can go further. Since $f^{(0)}$ is independent of both τ_0 and τ_1, equation (20) shows that the same applies to $\rho^{(0)}$. Equation (21) governs $\rho^{(1)}$. The inhomogeneous term is $\hat{\mathcal{E}} \cdot \hat{u}(\partial/\partial u)\rho^{(0)}$, which is a constant on the τ_1 as well as on the τ_0 scale. Moreover, the initial condition is $\rho^{(1)} = 0$, since it is $\rho^{(0)}$ that carries the entire initial condition. Thus, there is no τ_1 dependence anywhere in the equation governing $\rho^{(1)}$ and, consequently, $f^{(1)}$ must be *independent* of τ_1. Nevertheless, $\rho^{(1)}$ and $f^{(1)}$ do depend on τ_0.

It is the asymmetric term $f^{(1)}$ that carries the current, and the constant current on the τ_1-scale is that characteristic of the linear transport regime.

3.3. *Second order*

We shall now demonstrate that, to second order, we get a closed equation for $f^{(0)}$ on the τ_2-scale. Since $f^{(0)}$ is a constant on the τ_0- and τ_1-scales, the true initial state should also be used as initial data for this closed equation on the τ_2-scale. Having determined $f^{(0)}$, one can easily calculate $f^{(1)}$, from which the current follows on all three time scales.

To second order the Liouville equation reads

$$\frac{\partial\rho^{(0)}}{\partial\tau_2} - \hat{\mathcal{E}} \cdot \frac{\partial\rho^{(1)}}{\partial\boldsymbol{u}} + \left(\frac{\partial}{\partial\tau_0} + \alpha\boldsymbol{u} \cdot \frac{\partial}{\partial\boldsymbol{x}} + \frac{\partial}{\partial\phi}\right)\rho^{(2)} = 2\alpha\rho \sum_{j=1}^{N} T(e, j)\rho^{(2)} \tag{24}$$

where we used the fact that $\rho^{(1)}$ is independent of τ_1. Once again, integrate over $\boldsymbol{x}, 1, \ldots, N$ and act with the projector P on (24) to get (use equation (5) and, furthermore, remember that $f^{(0)}$ is rotationally invariant!):

$$\frac{\partial f^{(0)}}{\partial\tau_2} - P\left(\hat{\mathcal{E}} \cdot \frac{\partial}{\partial\boldsymbol{u}} Qf^{(1)}\right) + \frac{\partial Pf^{(2)}}{\partial\tau_0} = 0. \tag{25}$$

Here we used the assumption of spatial homogeneity, and the fact that only the complement of $Pf^{(1)}$, namely $Qf^{(1)} \equiv (1 - P)f^{(1)}$, survives the final averaging in the second term. The requirement that $f^{(2)}$ contain no secular terms when $\tau_0 \to \infty$ amounts to setting $\lim_{\tau_0\to\infty}(\partial/\partial\tau_0)Pf^{(2)} = 0$ in (25). Thus

$$\frac{\partial f^{(0)}}{\partial\tau_2} = \lim_{\tau_0\to\infty} P\left(\hat{\mathcal{E}} \cdot \frac{\partial}{\partial\boldsymbol{u}} Qf^{(1)}\right). \tag{26}$$

In order to make this a closed equation for $f^{(0)}$ on the τ_2-scale, we need the formal solution for $f^{(1)}$ on the τ_0-scale, in the limit $\tau_0 \to \infty$. The details involved in constructing this formal solution are a distraction from the multiple time scale line of thought and are, therefore, relegated to the appendix. The final result is that $f^{(0)}(u)$ obeys the equation

$$\frac{\partial f^{(0)}(u, \tau_2)}{\partial\tau_2} = \frac{1}{u}\frac{\partial}{\partial u}\mathcal{D}_1(u)\frac{1}{u}\frac{\partial}{\partial u} f^{(0)}(u, \tau_2). \tag{27}$$

Here $\mathcal{D}_1 = (\omega/v_0^2)D_1$ is the dimensionless, energy-dependent, diagonal part of the diffusion tensor ($D_1 = D_{xx} = D_{yy}$) or, in common terminology, the diffusion 'constant'.

4. Linear and nonlinear transport in the general Lorentz model

4.1. The diffusion equation

Introduction of the dimensionless kinetic energy, $\mathcal{K} = \frac{1}{2}u^2$, reduces (27) to the standard diffusion equation

$$\frac{\partial f^{(0)}(\mathcal{K}, \tau_2)}{\partial \tau_2} = \frac{\partial}{\partial \mathcal{K}} \mathcal{D}_1(\mathcal{K}) \frac{\partial f^{(0)}(\mathcal{K}, \tau_2)}{\partial \mathcal{K}}. \tag{28}$$

The relationship becomes even more transparent when one returns to the original physical variables

$$\tau_2 = \varepsilon^2 \tau = \left(\frac{\mathcal{E}}{\mathcal{B} v_0}\right)^2 \frac{e\mathcal{B}}{m} t \qquad \mathcal{D}_1 = \frac{e\mathcal{B}}{m v_0^2} D_1 \qquad \mathcal{K} = \frac{1}{2}u^2 = \frac{v^2}{2v_0^2} = \frac{K}{m v_0^2} \tag{29}$$

with the result, when superscripts on the distribution function are deleted,

$$\frac{\partial f(K, t)}{\partial t} = (e\mathcal{E})^2 \frac{\partial}{\partial K} D_1(K) \frac{\partial f(K, t)}{\partial K}. \tag{30}$$

Before discussing this equation we note that, in retrospect, we could have written it down directly, based on a physical argument at the phenomenological level. Let the initial state be one with kinetic energy $K = K_0 = \frac{1}{2} m v_0^2$, and with precisely given position $y = y_0$ on the y-axis, along which the electric field is oriented, but with a constant distribution with respect to x. As a function of time, the particles will undergo a diffusion process in the y-direction from this initial state. The diffusion constant depends on the constant parameters of the model and, in addition, on the kinetic energy of the moving particles. This kinetic energy is directly influenced by the electric, but not by the magnetic field. Since no inelastic processes are allowed in the model, the total energy, $K + e\mathcal{E}(y - y_0) = K_0$, is conserved. Thus, one-dimensional diffusion in the y-direction is governed by

$$\frac{\partial f(y, t)}{\partial t} = \frac{\partial}{\partial y} D_1[K_0 - e\mathcal{E}(y - y_0)] \frac{\partial f(y, t)}{\partial y}. \tag{31}$$

Clearly, equations (30) and (31) are one and the same. Even though the physical argument leading to (31) is direct and transparent, it is purely phenomenological and does not carry all the information inherent in our systematic derivation of (30). However, this direct argument serves as a welcome check on the systematic derivation.

From equation (30) a general result on the time development of the average kinetic energy follows immediately. Multiply (30) by K, integrate, and do two partial integrations on the right-hand side to get

$$\frac{\mathrm{d}\langle K \rangle}{\mathrm{d}t} = (e\mathcal{E})^2 \left\langle \frac{\mathrm{d}D_1(K)}{\mathrm{d}K} \right\rangle. \tag{32}$$

The evolution in (32) is governed by the *square* of the weak electric field! This is directly related to the fact that it occurs on the τ_2 time scale. Since the diffusion 'constant' quite generally is a monotonically increasing function of the kinetic energy, equation (32) demonstrates that its average will slowly increase with time.

4.2. The current density

The rotationally invariant part of the distribution function is constant on the τ_0- and τ_1-scales, and becomes interesting only on the τ_2-scale. The average current density, $\langle \boldsymbol{j} \rangle = n(-e)\langle \boldsymbol{v} \rangle$, on the other hand, has different and characteristic behaviour on all three scales.

On the τ_0-scale we start from an equilibrium distribution, i.e. from zero current density. The details of the transient τ_0-behaviour are, in general, hard to calculate systematically, although progress should be possible close to the Grad limit. At this point we shall be content with referring to (A10), which formally gives the current on the initial time scale.

We shall concentrate on the τ_1- and τ_2-scales here. From (A12), after reintroduction of unscaled variables, one has

$$\langle \boldsymbol{j} \rangle = n(-e)\langle \boldsymbol{v} \rangle = ne^2 \left[\hat{y} \left\langle \frac{\mathrm{d}D_1}{\mathrm{d}K} \right\rangle - \hat{x} \left\langle \frac{\mathrm{d}D_2}{\mathrm{d}K} \right\rangle \right] \mathcal{E}. \tag{33}$$

On the τ_1-scale the distribution function is still the initial Maxwellian, $f^{(0)} \sim \exp(-\beta K)$. Insertion into (33) and one partial integration gives

$$\langle \boldsymbol{j} \rangle_{\mathrm{eq}} = ne^2 \beta \left[\hat{y} \langle D_1 \rangle_{\mathrm{eq}} - \hat{x} \langle D_2 \rangle_{\mathrm{eq}} \right] \mathcal{E} \tag{34}$$

in agreement with the Nernst–Einstein relation (for linear transport!), which gives the conductivity tensor as $ne^2\beta$ times the diffusion tensor.

On the τ_2-scale the results are formally similar in that (33) still holds true, but $f^{(0)}$ no longer equals the initial Maxwellian, and must rather be determined from the solution of the diffusion equation (30). That is, transport is no longer linear. However, as follows from (33), with $\langle \cdots \rangle_{\mathrm{eq}}$ replaced by the average with respect to the time dependent distribution resulting from (30), nonlinear transport is *also* governed by the time integral of the velocity correlation functions!

5. Explicit results for low density

So far the results apply to *any* 2D Lorentz model, with straightforward generalizations to 3D. In particular, no assumption was introduced restricting our results to low density. However, it is instructive to specialize further to allow for more explicit results. What is needed is expressions for the diffusion tensor in a 2D Lorentz model in a magnetic field. At this point, then, we adopt results that, strictly speaking, only apply in the Grad limit. For magnetotransport in 2D, the standard Boltzmann equation is *not* correct. The basic reason [2] for this is that there is a finite probability, $P_0 = \exp(-2\pi R_c/\Lambda)$, (with R_c the cyclotron radius and Λ the mean-free path) that an electron completes an entire cyclotron orbit without scattering†. Thus, a fraction P_0 of the electrons are 'circling', they never collide. The remaining electrons are 'wandering', they collide (in the Grad limit, and in the course of time) with infinitely many different scatterers. However, for the same reason that circling eleyctrons exist, the wandering electrons can recollide many times with the same scatterer before proceeding to the next. As a result, the correct 'generalized Boltzmann equation' (GBE), has a *non-Markovian* structure [2]:

$$\left(\frac{\partial}{\partial t} + \omega \frac{\partial}{\partial \phi} \right) f^{\mathrm{G}}(\phi, t) = \sum_{s=0}^{[t/T]} P_0^s \nu \int_{-\pi}^{\pi} \mathrm{d}\psi \; g(\psi) \big[f^{\mathrm{G}}(\phi - (s+1)\psi, t - sT) - f^{\mathrm{G}}(\phi - s\psi, t - sT) \big]. \tag{35}$$

Here $\nu T = 2\pi R_c/\Lambda$, ν is the collision frequency, $T = 2\pi/\omega$ the cyclotron period, and $[t/T]$ denotes the integral part of t/T. The function $g(\psi) = \sigma(\psi)/\int \mathrm{d}\psi\, \sigma(\psi)$ is the dimensionless differential scattering cross section. The superscript G on f^{G} refers to the

† *Proof.* The scatterers of radius a are distributed randomly in the plane with density n. For a cyclotron orbit with radius R_c to be completed without scatterings, the area $A_c = 4\pi a R_c$ between radii $R_c - a$ and $R_c + a$ must be free of scattering centres. The probability that this is so is $P_0 = \exp(-nA_c) = \exp(-2\pi R_c/\Lambda)$. □

subtlety that, for $t \geqslant T$, equation (35) applies to the wandering electrons only, not to the circling ones, whereas for $t < T$ (when only the first, standard Boltzmann, term on the right-hand side survives) the distinction between circling and wandering electrons has not yet been fully made, and the GBE describes *all* electrons. The sth (non-Markovian!) term on the right-hand side describes the sth recollision with the same scatterer. For an intuitive derivation and further discussion of (35), we refer the reader to [2]. A controlled derivation of (35) from (3) would clearly be desirable, but remains an open problem.

What is needed in the present context is the diffusion tensor that follows from the solution of the initial value problem posed by equation (35), through the standard Kubo formula (A6), (A11), (see [2]). Writing it as a complex quantity, D, with $D_1 = \mathrm{Re}\, D$ and $D_2 = \mathrm{Im}\, D$, we refer the reader to [2] for the following result, valid for hard disc scatterers†

$$D = \frac{1}{2}v^2\left[\frac{P_0}{-\mathrm{i}\omega} + \frac{(1-P_0)\tau_D(\sqrt{P_0})}{1-\mathrm{i}\omega\tau_D(\sqrt{P_0})}\right]$$

$$\tau_D(p) = \nu^{-1}\left[1 - \frac{1-p^2}{2p^2}\left(\frac{1-p^2}{2p}\ln\frac{1+p}{1-p} - 1\right)\right]^{-1} \equiv \nu^{-1}h(p). \qquad (36)$$

The function $h(p)$ is monotonic‡ on the interval $0 \leqslant p < 1$, corresponding to the magnetic field interval $0 \leqslant \mathcal{B} < \infty$, and varies from $h(0) = \frac{3}{4}$ to $h(1) = 1$. Since $\Lambda = v/\nu$ is an energy-independent constant, we can introduce a dimensionless kinetic energy

$$k = (\nu/\omega)^2 = \frac{2K}{m\Lambda^2\omega^2} \qquad (37)$$

and write equation (36) in the form

$$D(k) = \frac{\omega\Lambda^2}{2}\sqrt{k}\left[\frac{e^{-2\pi\sqrt{k}}}{-\mathrm{i}/\sqrt{k}} + \frac{(1-e^{-2\pi\sqrt{k}})h(e^{-\pi\sqrt{k}})}{1-\mathrm{i}h(e^{-\pi\sqrt{k}})/\sqrt{k}}\right]. \qquad (38)$$

5.1. The diffusion equation

First let us consider the diagonal part D_1 which governs the time evolution on the τ_2-scale, through the diffusion equation (30):

$$D_1 = \frac{\omega\Lambda^2}{2}\frac{(1-e^{-2\pi\sqrt{k}})h(e^{-\pi\sqrt{k}})\sqrt{k}}{1+h^2(e^{-\pi\sqrt{k}})/k}. \qquad (39)$$

Clearly, an analytic solution of the general initial problem posed by the diffusion equation (30), with the complicated diffusion constant $D_1(K)$ given by (39) via relation (37), is out of the question. Here we shall be content with commenting on the two extremes, namely when $k \ll 1$, and when $k \gg 1$.

If the magnetic field is sufficiently strong so that in the initial equilibrium state, $\omega \gg \sqrt{2/(m\beta)}\Lambda^{-1}$ (with $\beta = (k_B T)^{-1}$, the inverse initial temperature), one has $k \ll 1$, and D_1 simplifies to

$$D_1 \simeq \pi\omega\Lambda^2 k^2 = \frac{4\pi}{m^2\Lambda^2\omega^3}K^2 \qquad (40)$$

† The first, purely imaginary, term is the contribution to D from circling electrons for $t > T$ (not described by (35)). It was erroneously overlooked in [2](a), but has been included in [2](b).

‡ The small-p expansion of $h(p)$ reads $h(p) = (3/4)[1 + p^2/5 + 12p^4/175 + \cdots]$, and converges rapidly for $p \leqslant 1$.

From equation (32) it then follows that, as long as the condition $k \ll 1$ holds, the initial growth of the mean kinetic energy is exponential:

$$\langle K \rangle \simeq \beta^{-1} \exp\left[\frac{8\pi}{\omega}\left(\frac{\mathcal{E}}{\mathcal{B}}\right)^2 \frac{t}{\Lambda^2}\right] = \beta^{-1} \exp\left[4\left(\frac{v_{\rm d}T}{\Lambda}\right)^2 \frac{t}{T}\right] \tag{41}$$

with the drift velocity $v_{\rm d} = \mathcal{E}/\mathcal{B}$ and the cyclotron period $T = 2\pi/\omega$.

As the average kinetic energy grows in time, the non-Markovian effects of the GBE will decay together with $e^{-\pi\sqrt{k}}$. Finally, the kinetic energy will be sufficiently large that the cyclotron paths between collisions closely resemble straight lines. In this final stage, the magnetic field has effectively been switched off, $k \gg 1$, and

$$D_1 \simeq \frac{3\omega\Lambda^2}{8}\sqrt{k} = \frac{3\sqrt{2}\Lambda}{8\sqrt{m}}\sqrt{K}. \tag{42}$$

Since for large kinetic energies the diffusion constant depends on the energy as a simple power law, the asymptotic scaling solution to the diffusion equation can easily be found:

$$f(K,t) \simeq \frac{c}{t^{2/3}} \exp[-K^{3/2}/(\alpha t)] \tag{43}$$

where $\alpha = 27\sqrt{2}\Lambda(e\mathcal{E})^2/(32\sqrt{m})$ and $c = 3m/[4\pi\alpha^{2/3}\Gamma(2/3)]$ is a normalization constant. The immmediate consequences are the well known asymptotic results [4, 5] $\langle K \rangle \sim \mathcal{E}^{4/3}t^{2/3}$ and $\langle v \rangle \sim \mathcal{E}^{2/3}t^{1/3}$.

5.2. *The current density*

The two components of the current density follow from (33). We first consider the component parallel to the electric field, i.e. $\langle j_y \rangle$. In the case that $k \ll 1$, equations (40) and (41) give the initial exponential growth on the τ_2-scale as

$$\langle j_y \rangle \simeq \mathcal{E}\frac{ne^2}{\beta}\frac{8\pi}{m^2\Lambda^2\omega^3} \exp\left[4\left(\frac{v_{\rm d}T}{\Lambda}\right)^2 \frac{t}{T}\right] \tag{44}$$

starting from the linear response value $\langle j_y \rangle_{\rm eq}$ of (34) for the case $k \ll 1$. When sufficient time has elapsed so that the opposite extreme, $k \gg 1$, has been reached, the diffusion constant $D_1 \sim \sqrt{K}$, so that $\mathrm{d}D_1/\mathrm{d}K \sim (\sqrt{K})^{-1}$. That is, we retrieve the familiar result [4, 5] $\langle j_y \rangle \sim \mathcal{E}^{1/3}t^{-1/3}$.

Turning now to the component of the current density perpendicular to the electric field, $\langle j_x \rangle = -ne^2\mathcal{E}\langle \mathrm{d}D_2/\mathrm{d}K \rangle$, equation (38) gives

$$D_2 = \frac{\omega\Lambda^2}{2}\sqrt{k}\left[\sqrt{k}e^{-2\pi\sqrt{k}} + \frac{(1-e^{-2\pi\sqrt{k}})h^2(e^{-\pi\sqrt{k}})/\sqrt{k}}{1+h^2(e^{-\pi\sqrt{k}})/k}\right]. \tag{45}$$

We again consider the case when the $\mathcal{B}$-field is sufficiently strong that, in the initial equilibrium state, $k \ll 1$. Retaining the two lowest terms we find

$$D_2 \simeq \frac{1}{e\mathcal{B}}K - \frac{4\sqrt{2}\pi m^{3/2}}{\Lambda^3 e^4 \mathcal{B}^4}K^{5/2} + \cdots. \tag{46}$$

The first term gives

$$\langle j_x \rangle \simeq -\frac{ne\mathcal{E}}{\mathcal{B}} = -nev_{\rm d} \tag{47}$$

as it should. The second term, $\sim \langle K^{3/2} \rangle$, reduces the magnitude of the current, exponentially fast at first.

The decay of the transverse current, when $k \gg 1$, is governed by

$$D_2 \simeq \frac{\omega\Lambda^2}{2} g^2(0) \left[1 - \frac{g^2(0)}{k} + \cdots\right] = \frac{9\omega\Lambda^2}{32}\left[1 - \frac{9m\omega^2\Lambda^2}{32K} + \cdots\right]. \quad (48)$$

From this the final decay follows as

$$\langle j_x \rangle \simeq -\mathcal{E} n e^2 3^4 2^{-10} m \Lambda^4 \omega^3 \langle K^{-2} \rangle \sim -\mathcal{B}^3 \mathcal{E}^{-5/3} t^{-4/3}. \quad (49)$$

In this section we have only given the leading terms of the various asymptotic results. In order to determine the corrections to the leading asymptotic terms, and their dependence on, e.g., the initial temperature, connection problems analagous to those studied by Olaussen and Hemmer [5] have to be solved.

6. Concluding remarks

This paper contains two general results on Lorentz models: The energy diffusion equation (30), and the expression for the current density (33). No restriction to small densities of scatterers was used in the derivation of these results. Technically, it is true that the derivation was based on a specific model for the scatterers, namely hard discs. However, this restriction can easily be lifted. Also, dimensionality plays no essential role here, except for the distinction between motion parallel and transverse to the magnetic field. We claim that our results for linear and nonlinear transport have, for the dissipationless Lorentz model, the same status as the Kubo formulae for linear transport coefficients in the general case.

This work demonstrates once again the efficiency of the systematic multiple time scale method. In our case it provides a clean separation between three time scales governed by different physics. As the explicit results of section 5 show, the τ_2-scale can be subdivided into two: the initial period in which subtle magnetic field effects are important, and the final one dominated by electric field effects only.

It is interesting to note that, despite the fact that the dimensionless electric field was assumed to be smaller than all the other parameters, it dominates after a sufficiently long time has elapsed.

Nevertheless, the entire argument of the present paper rests on the electric field being very small. Increasing the electric field, we expect to find a nonlinear crossover from physics described (at low densities) by the generalized Boltzmann equation, to physics governed by the Boltzmann equation proper. The quantitative understanding of this crossover remains an open problem. In this connection one should also note that even the classical mechanics of an electron and a *single* scatterer in crossed electric and magnetic fields is a subtle problem [11] in nonlinear mechanics. For a full discussion, the machinery of Kolmogorov–Arnold–Moser theory (see, for example, [12]) is needed. A surprising qualitative result is that, even in finite (but weak) electric fields, an electron can be trapped by a repulsive scatterer [11]. In the present paper there is no trace of these subtle effects. The basic reason for this is simple: in a weak electric field, we keep the leading term only. This implies that basic to the calculation of the diffusion tensor (the ‘coefficient’ of the leading term) is mechanics with *no* electric field present. In addition to this asymptotic fact, averaging over initial conditions would tend to smear out subtle aspects of the nonlinear mechanics in finite electric and magnetic fields.

Evidently, beyond the small-$\mathcal{E}$ asymptotics studied in this paper, a number of intriguing questions remain to be clarified.

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Appendix. Formal solution to first order

Here we formally solve for $f^{(1)}$ on the τ_0, τ_1 and τ_2 time scales. We first demonstrate that a closed equation for $f^{(0)}$ on the τ_2-scale follows. That is, we fill in the missing steps between (26) and (27). We then turn to the current density and determine it formally on all three time scales.

We begin by writing down the formal solution of (21), using (20) and the fact that $\rho^{(0)}$ is independent of τ_0 and of τ_1

$$\rho^{(1)}(e, 1, \ldots, N; \tau_0, \tau_2, \ldots) = \frac{1}{A}\int_0^{\tau_0} \mathrm{d}\tau' \, \exp\left\{-(\tau_0 - \tau')\left[\alpha \boldsymbol{u}\cdot\frac{\partial}{\partial \boldsymbol{x}} + \frac{\partial}{\partial \phi} - 2\alpha\eta\sum_{j=1}^{N} T(e, j)\right]\right\}$$
$$\times \hat{\mathcal{E}}\cdot\hat{u}\,\frac{\partial f^{(0)}(u; \tau_2, \ldots)}{\partial u}\prod_{j=1}^{N}\left\{\frac{\theta(|\boldsymbol{x} - \boldsymbol{X}_j| - 2\eta)}{A - 4\pi\eta^2}\right\}. \tag{A1}$$

Apply to both sides of (A1) the operator $\lim_\infty \int \mathrm{d}\boldsymbol{x} \int \mathrm{d}1 \cdots \mathrm{d}N$, introduce (temporarily) the variable $\tau'' = \tau_0 - \tau'$, and note that $\hat{\mathcal{E}}(\partial f^{(0)}/\partial u)\prod\{\cdots\}$ in (A1) is not affected by the $(\tau_0 - \tau')$ operator. This gives

$$f^{(1)}(\boldsymbol{u}; \tau_0, \tau_1, \ldots) = \frac{\partial f^{(0)}}{\partial u}\hat{\mathcal{E}}\cdot\int_0^{\tau_0}\mathrm{d}\tau'' \lim_{\infty}\frac{1}{A}\int \mathrm{d}\boldsymbol{x}\int \mathrm{d}1\cdots\mathrm{d}N \prod\{\cdots\}\exp\left(-\tau''[\cdots]\right)\hat{u}$$
$$= \frac{\partial f^{(0)}}{\partial u}\hat{\mathcal{E}}\cdot\int_0^{\tau_0}\mathrm{d}\tau'' \lim_{\infty}\frac{1}{A}\int \mathrm{d}\boldsymbol{x}\int \mathrm{d}1\cdots\mathrm{d}N \prod\{\cdots\}\hat{u}(-\tau''; \boldsymbol{x}, 1, \ldots, N)|_{\hat{u}(0)=\hat{u}}$$
$$= \frac{\partial f^{(0)}}{\partial u}\hat{\mathcal{E}}\cdot\int_0^{\tau_0}\mathrm{d}\tau'\langle\hat{u}(-\tau')\rangle_{\hat{u}(0)=\hat{u}}. \tag{A2}$$

Here $\langle\cdots\rangle$ denotes the average, in the thermodynamic limit, over the positions of the scatterers and the electron.

A.1. The diffusion equation

We first use equation (A2) to derive a closed equation for $f^{(0)}$ on the τ_2-scale by inserting the resulting $f^{(1)}$, in the limit $\tau_0 \to \infty$, into (26). Clearly, the angular average with repect to $\hat{u}$ of the integral in (A2) vanishes, i.e. $Q = 1 - P$ will project it onto itself. We can therefore ignore this projector in (26). We then use the identity (we have chosen $\hat{\mathcal{E}} = \hat{y}$)

$$\hat{\mathcal{E}}\cdot\frac{\partial}{\partial \boldsymbol{u}} = \sin\phi\frac{\partial}{\partial u} + \frac{\cos\phi}{u}\frac{\partial}{\partial\phi} \tag{A3}$$

to write

$$\frac{\partial f^{(0)}}{\partial \tau_2} = \lim_{\tau_0\to\infty} P\left(\hat{\mathcal{E}}\cdot\frac{\partial}{\partial \boldsymbol{u}} Q f^{(1)}\right)$$

$$= P\left\{\int_0^\infty \mathrm{d}\tau' \left[\frac{\partial}{\partial u}\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}\phi\ \langle \sin\phi \sin\phi(-\tau')\rangle_{\phi(0)=\phi}\right.\right.$$

$$\left.\left. + \frac{1}{u}\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}\phi\ \langle \cos\phi \frac{\partial}{\partial\phi}\sin\phi(-\tau')\rangle_{\phi(0)=\phi}\right]\frac{\partial f^{(0)}}{\partial u}\right\}. \quad \text{(A4)}$$

Since ϕ and $\langle\phi(-\tau')\rangle_{\phi(0)=\phi}$ are proportional to one another, the second average is over a product of cosines and, by symmetry, equals the average over the product of sines. We can therefore add them and divide by 2 to get $\frac{1}{2}\langle\cos[\phi - \phi(-\tau')]\rangle_{\phi(0)=\phi}$. In other words, we have deduced that

$$\frac{\partial f^{(0)}}{\partial \tau_2} = P\left\{\left(\frac{\partial}{\partial u} + \frac{1}{u}\right)\frac{1}{2}\int_0^\infty \mathrm{d}\tau'\ \langle \hat{u}\cdot\hat{u}(-\tau')\rangle_{\hat{u}(0)=\hat{u}}\frac{\partial f^{(0)}}{\partial u}\right\}. \quad \text{(A5)}$$

Clearly $\{\cdots\}$ in (A5) is already rotationally invariant, so P projects it onto itself. Finally, we note that the diagonal part of the diffusion tensor, in physical variables, reads

$$D_1 = \tfrac{1}{2}\int_0^\infty \mathrm{d}t\ \langle \boldsymbol{v}(t)\cdot\boldsymbol{v}(0)\rangle = \frac{v_0^2 u^2}{\omega}\frac{1}{2}\int_0^\infty \mathrm{d}\tau'\ \langle \hat{u}\cdot\hat{u}(-\tau')\rangle \equiv \frac{v_0^2}{\omega}\mathcal{D}_1 \quad \text{(A6)}$$

where time translation invariance of the equilibrium average was used. In terms of $\mathcal{D}_1/u^2$, equation (A5) becomes

$$\frac{\partial f^{(0)}}{\partial \tau_2} = \left(\frac{\partial}{\partial u} + \frac{1}{u}\right)\frac{\mathcal{D}_1}{u^2}\frac{\partial f^{(0)}}{\partial u} = \frac{1}{u}\frac{\partial}{\partial u}\mathcal{D}_1\frac{1}{u}\frac{\partial}{\partial u}f^{(0)}. \quad \text{(A7)}$$

Introducing the dimensionless kinetic energy $\mathcal{K} = \frac{1}{2}u^2$, we can finally write

$$\frac{\partial f^{(0)}}{\partial \tau_2} = \frac{\partial}{\partial \mathcal{K}}\mathcal{D}_1(\mathcal{K})\frac{\partial f^{(0)}}{\partial \mathcal{K}}. \quad \text{(A8)}$$

No assumption on the density of scatterers were used in the above derivation. In fact, it can be read as containing a (complicated) rederivation of the Einstein/Kubo formula for the energy dependent diffusion 'constant'.

A.2. The current density

Next we use equation (A2) to calculate an expression for the average velocity, i.e. the current, on all time scales. The mean velocity is given by

$$\langle \boldsymbol{v}\rangle = v_0\langle \boldsymbol{u}\rangle = v_0\int \mathrm{d}\boldsymbol{u}\ \boldsymbol{u}\left(f^{(0)} + \varepsilon f^{(1)} + \cdots\right) = v_0\varepsilon\int \mathrm{d}\boldsymbol{u}\ \boldsymbol{u} f^{(1)} \quad \text{(A9)}$$

when contributions of higher order than ε are neglected. Splitting $\boldsymbol{u}$ in $\hat{\mathcal{E}}$-, i.e. $\hat{y}$- and $\hat{x}$-, components, equation (A2) can be used to rewrite (A9) as

$$\langle \boldsymbol{v}\rangle = v_0\varepsilon\int \mathrm{d}\boldsymbol{u}\ u^{-1}\left(\hat{y}\int_0^{\tau_0}\mathrm{d}\tau\ \langle u_y(-\tau)u_y(0)\rangle + \hat{x}\int_0^{\tau_0}\mathrm{d}\tau\ \langle u_y(-\tau)u_x(0)\rangle\right)\frac{\partial f^{(0)}}{\partial u}. \quad \text{(A10)}$$

In order to determine the τ_0-dependence of $\langle \boldsymbol{v}\rangle$, equation (A10) shows that the solution of the initial value problem of the corresponding kinetic theory is needed. In the Grad limit, the appropriate kinetic equation is the generalized Boltzmann equation [2]. We do not pause to give details here.

On the τ_1- and τ_2-scales, we need the limiting value of (A10) as $\tau_0 \to \infty$. The two time integrals appearing in (A10) are closely related to the (rescaled) diagonal and off-diagonal terms, respectively, of the diffusion tensor

$$\int_0^\infty \mathrm{d}\tau \ \langle u_y(-\tau) u_y(0) \rangle = \mathcal{D}_1 = \frac{\omega D_1}{v_0^2} \qquad \int_0^\infty \mathrm{d}\tau \ \langle u_y(-\tau) u_x(0) \rangle = -\mathcal{D}_2 = -\frac{\omega D_2}{v_0^2}. \tag{A11}$$

Here $D_1 = D_{yy} = D_{xx}$ and $D_2 = D_{yx} = -D_{xy}$. It is again convenient to introduce the dimensionless kinetic energy $\mathcal{K} = \frac{1}{2}u^2$. Note that the normalization $\int \mathrm{d}\boldsymbol{u}\, f(u) = 1$ implies $2\pi \int_0^\infty \mathrm{d}\mathcal{K}\, f(\mathcal{K}) = 1$ in our two-dimensional case. Using this and (A11) in (A10) gives, by partial integration,

$$\begin{aligned} \langle \boldsymbol{v} \rangle &= v_0 \varepsilon \left[\hat{y} 2\pi \int \mathrm{d}\mathcal{K}\ \mathcal{D}_1(\mathcal{K}) \frac{\partial f^{(0)}}{\partial \mathcal{K}} - \hat{x} 2\pi \int \mathrm{d}\mathcal{K}\ \mathcal{D}_2(\mathcal{K}) \frac{\partial f^{(0)}}{\partial \mathcal{K}} \right] \\ &= -\hat{y} v_0 \varepsilon \left\langle \frac{\mathrm{d}\mathcal{D}_1}{\mathrm{d}\mathcal{K}} \right\rangle + \hat{x} v_0 \varepsilon \left\langle \frac{\mathrm{d}\mathcal{D}_2}{\mathrm{d}\mathcal{K}} \right\rangle. \end{aligned} \tag{A12}$$

In (A12) $\langle \cdots \rangle$ implies averaging with respect to $f^{(0)}$.

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